THE

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THE ECCENTRIC CIRCLE OF BOSCOVICH.

II. CONSIDERATION of the special case in which the centre of the eccentric circle lies on the straight line whose points of intersection with the conic are required.

In what follows it will be convenient to abandon the system of lettering adopted by Boscovich which we have hitherto followed, and as far as possible to use that common to modern text books.

1. *Having given the focus, directrix, and eccentricity of a conic, to find its intersection with a given straight line.*

Let the given straight line HK (Fig. 1) cut the directrix in Z. Take any point O on HK and draw OL perpendicular to the directrix. With centre O and radius eOL describe a circle cutting SZ in Q, q. Through S draw SP, Sp parallel to OQ, Oq meeting HK in P, p. Then P, p are the required points of intersection.

Draw PM perpendicular to the directrix.

Then $SP : PZ :: OQ : OZ$

And $PZ : PM :: OZ : OL$ } by similar triangles.

$\therefore SP : PM :: OQ : OL.$

$\therefore SP = e\text{PM}.$

$\therefore$ P lies on the conic.

Similarly p lies on the conic.

Conversely it may be shown that if a straight line HK, meeting the directrix in Z, cut the conic in P, p and from a point O on HK straight lines OQ, Oq are drawn parallel to SP, Sp to meet SZ in Q, q—

$$\text{OQ} = e\text{OL} = \text{O}q,$$

when OL is perpendicular to the directrix.

Hence (i.) SZ is equally inclined to SP, Sp; (ii.) HK cannot cut the conic in more than two points.

2. *To find any number of points on a conic whose focus, directrix, and eccentricity are given.*

From any point O (Fig. 1) draw OL perpendicular to the directrix. With centre O and radius eOL describe a circle. Draw any radius OQ. Join SQ, cutting the directrix in Z. Draw SP parallel to OQ, cutting OZ in P. Then by the above demonstration P is a point on the conic.

It is worthy of remark that—

(i.) $SP : OQ :: SZ : ZQ.$

(ii.) The ratio SZ : ZQ is finite except where Q coincides with Z.

Hence when $e < 1$, SP always remains finite.

„ $e = 1$, SP becomes indefinitely great only as its direction approaches to that of OZ.

„ $e > 1$, SP becomes indefinitely great when its direction approaches that of either ON, On, N, n being the points where the circle cuts the directrix.[1]

Returning to proposition 1 it is clear (i.) that if Q, q coincide so also do P, p and conversely. Hence if SQ touches the circle, OP touches the conic, and conversely.

(ii.) If the tangent at P meet the directrix in Z then ZSP is a right angle. Hence the solution of the following problem :—

3. *From a given point to draw a pair of tangents to a conic whose focus, directrix, and eccentricity are given.*

From the given point O (Fig. 2) draw OL perpendicular to the directrix. With centre O and radius eOL describe a circle, and from S draw SQ touching the circle in Q and cutting the directrix in Z. Then OZ shall be a tangent to the conic. Draw SP parallel to OQ, meeting OZ in P, and draw PM perpendicular to the directrix.

Then $SP : PZ :: OQ : OZ.$

And $PZ : PM :: OZ : OL.$

$\therefore SP : PM :: OQ : OL.$

$\therefore SP = e\text{PM}.$

$\therefore$ P lies on the conic.

And since angle ZSP = angle OQZ, which is right, therefore OP is a tangent.

Similarly a second tangent can be drawn.

Conversely, if from a point O on the tangent at P to a conic which meets the directrix in Z, perpendiculars OQ, OL be drawn to SZ and the directrix, then $\text{OQ} = e\text{OL}$.

It easily follows that if OP, Op are tangents to a conic from O, then angle OSP = angle OSp. (See *Haslam and Edwards*, p. 9.)

[1] Compare I. 3. In the diagrams 3, 4, 5 on the sheet supplied with No. 1 of the *Gazette* the points N, n should be supplied.

III. General treatment of Boscovich's method as one of transformation.[1]

1. *Def.—Let AB (Fig. 3) be a given fixed straight line, S and O two given fixed points, SP and OQ any pair of parallels through S and O, then if OP, SQ intersect on AB, the points P and Q are said to "correspond" to each other.*

Note that if Z be the point of intersection of OP, SQ—

$$SP : OQ :: SZ : QZ.$$

Hence if Q approaches indefinitely near to AB, P recedes to infinity.

Similarly if P approaches indefinitely near to AB, Q recedes to infinity.

From this property the line AB will be called *the vanishing line* and be denoted by *v.l.*

2. *Prop.—If P describes a straight line PH meeting the vanishing line AB in H, then Q describes a straight line LQ parallel to SH.*

Let the parallel to PH (Fig. 3) through O meet AB in L. Then L is a fixed point. Join LQ.

By similar triangles LZ : ZO : : HZ : ZP.

And ZO : ZQ : : ZP : ZS,

$\therefore$ LZ : ZQ : : HZ : ZS.

And angle LZQ = angle HZS.

$\therefore$ angle ZLQ = angle ZHS.

$\therefore$ Q lies on a fixed straight line LQ parallel to SH.

Hence the following definition :—

3. *Corresponding straight lines are such that each is the locus of a point which corresponds to a point on the other.*

Instead of proceeding in the above order it is plain that—

(i.) We might have first given the following definition of corresponding straight lines :—

Let AB be a given fixed straight line, O and S any two given fixed points, HP, LQ any two lines drawn from points H and L on AB parallel to OL, SH; then the lines HP, LQ are said to "correspond" to each other.

(ii.) We might then have proved the following proposition :—

If a straight line PH pass through a fixed point P, then the corresponding straight line LQ passes through a fixed point Q such that SP and OQ are parallel.

(iii.) We might then have defined *corresponding points* as *the points of intersection of corresponding straight lines.*

The process of obtaining from a given straight line PH the straight line LQ which corresponds to it is called *reversion.* To distinguish between the original PH and LQ the derived line LQ is called the *reverse* of PH, while PH is called the *obverse* of LQ.[2] Note that the relationship is not strictly reciprocal unless O and S coincide.

4. *A given angle may be reverted into an angle of any given magnitude.*

Let the arms CB, CA (Fig. 4) of the given angle BCA meet the v.l. in a, b, and let S be any point on the arc aSb containing an angle of the given magnitude. Through O draw parallels to aC, bC meeting the v.l. in a', b'. Through a', b' draw $a'C'$, $b'C'$ parallel to Sa, Sb. Then angle $a'C'b'$, which is the reverse of angle aCb, has the given magnitude.

5. *A given triangle may be reverted into an equilateral triangle.*

Let the sides BC, CA, AB of the given triangle meet the v.l. in a, b, c (Fig. 4). Let S be the intersection of the circumcircles of the equilateral triangles described on ab, bc. Through O draw parallels to BC, CA, AB meeting the v.l. in a', b', c'. Through a', b', c' draw parallels B'C', C'A', A'B' to Sa, Sb, Sc. Then triangle A'B'C', which is the reverse of triangle ABC, is equilateral.

6. *Any quadrilateral may be reverted into* (i.) *a parallelogram;* (ii.) *a rectangle;* (iii.) *a square.*

(i.) Let AB, CD meet in E; BC, DA in F; then for all positions of S the reverse quadrilateral is a parallelogram, if we revert EF to infinity.

(ii.) If S is taken anywhere on the circle which has EF for a diameter, the reverse quadrilateral will be a right-angled parallelogram.

(iii.) Let the diagonals AC, DB meet EF in H, K, then if S be taken at one of the points of intersection of the circles whose diameters are EF, HK, the diagonals as well as the adjacent sides of the reverse quadrilateral will be at right angles. Hence it is a square.

7. *Tangents revert into tangents.*

Hence all properties of poles and polars revert.

8. *The cross ratio of a pencil is unaltered by reversion.*

Let the rays of a pencil whose vertex is P meet the v.l. in a, b, c, d, and the corresponding rays through the reverse point Q meet the v.l. in a', b', c', d'.

Then $$Q(a'b'c'd') = S(abcd),$$
$$= P(abcd).$$

9. *Hence the anharmonic property of a conic.* For by taking S at the focus and the directrix for v.l. a conic can be reverted into a circle.

Hence also *a conic reverts into a conic.*

10. If the polar of S be taken for vanishing line a conic reverts into a conic whose centre is O.

11. *A given conic reverts into an* (iii.) *ellipse,* (ii.) *a parabola,* (i.) *a hyperbola, according as the v.l.* (i.) *cuts,* (ii.) *touches,* (iii.) *does not meet the given conic.*

12. *If a chord of a conic subtend a right angle at a fixed point S on the conic, it always passes through a fixed point on the normal at S (Frégier's Theorem).*

The following neat proof is given by *Haslam and Edwards,* p. 101 :—

Let Pp, $P'p'$ (Fig. 5), two of the chords subtending right angles at S, intersect in E.

Take the polar of E for v.l.

Then the corresponding chords Qq, $Q'q'$ of the reverse conic intersect at F, which must be its centre since its polar is at infinity.

[1] In a paper on *The Discovery and the Geometrical Treatment of the Conic Sections,* read before the A. I. G. T. in 1884 and published in the Tenth General Report, Dr. Taylor strongly advocated the use of Boscovich's eccentric circle, and remarked on it as "one of the simplest introductions to homographic transformation in general."

[2] See Dr. Taylor's *Ancient and Modern Geometry of Conics.*

But QOq, Q′O′q′ are right angles.

Hence QF = OF = qF = Q′F = q′F.

∴ The reverse conic is a circle.

Hence the reverse Q″q″ of any other chord P″p″ subtending a right angle at S must pass through F, since Q″Oq″ is a right angle. Hence P″p″ must pass through E.

13. *Any conic can be reverted into a circle, any straight line which does not cut it being taken for the vanishing line.*

For the proof see *Haslam and Edwards*, p. 99.

IV. Connection between reversion and perspective projection.

It requires very little acquaintance on the part of the student with the method of perspective projection to enable him to discover the close analogy of the results obtained by the method of *reversion* and by that of *perspective projection*, the resemblance extending even to minute details of construction. Geometrical instinct naturally leads us to suppose that there must exist some close relationship between the two methods. We proceed to demonstrate its existence.

On the opposite side of the vanishing line AB to O (Fig. 6), and at an equal distance from it draw a straight line parallel to it. Let OP cut this parallel in K, and draw KU parallel to SQ to meet SP in U.

Then SU : SP : : ZK : ZP
: : OZ : ZP
: : OQ : SP,
∴ SU = OQ.

∴ The locus of U is congruent with that of Q.

But U is the perspective projection of P, S being the *vertex* and HK the *axis of perspective*.

Now there is no reason why O and S should not coincide, and then Boscovich's transformation will be absolutely identical with the perspective projection. All we have to do is to take our axis of perspective on the opposite side of the vanishing line to the vertex S and at an equal distance from it. The figure is worthy of some attention.

E. M. Langley.

GREEK GEOMETERS BEFORE EUCLID.

(*Continued from No.* 1.)

Pythagoras and the Italic School.

Pythagoras was born in Samos about 580 B.C. The statements regarding his masters are destitute of solid historical foundation, and so are the accounts of his travels. It is known, however, that he visited Egypt. He may have taught in Samos, but whether he did not obtain sufficient consideration among his fellow-citizens, or whether he wished to withdraw from the tyranny of Polycrates, he removed to Croton about 540 B.C. There he founded an association, less scientific than politico-religious, which eventually became the nucleus of the aristocratic party. About this time a democratic movement convulsed all the Greek States, and, in consequence, the Pythagorean school was dissolved, and its members dispersed. What happened to the head of the school is not known, whether he was killed in a tumult at Croton, or died a natural death in exile at Metapontum.

The ancient testimonies regarding Pythagoras and the Pythagoreans are not always in accord with each other, and it is impossible to distinguish between what is due to the master and what to his disciples. Eudemus states that Pythagoras raised geometry to the dignity of a science, that he gave to mathematics the name it now bears, and that the three proportions, arithmetical, geometrical, harmonical, were known to him. He also ascribes to the Pythagoreans the theorem that the sum of the three angles of any triangle is equal to two right angles (they proved it by drawing through the vertex a parallel to the base), and the closely related theorem that only three regular polygons (the equilateral triangle, the square, the regular hexagon) can fill up the space round a point. It may, however, be reasonably conjectured that the Egyptians were aware of the latter fact from experiment, and that Pythagoras or his disciples gave it a rigorous proof. The question of dividing the plane into regular congruent polygons is intimately connected with another discovery attributed to Pythagoras, that of the construction of the five regular polyhedrons. Of these *cosmic figures*, as Proclus calls them, the cube, the tetrahedron, and the octahedron were known to the Egyptians. To the Pythagoreans were due the problems concerning the application (parabola), the application in excess (hyperbola), and the application in defect (ellipse) of areas. What these problems are may be seen by consulting Euclid I. 44 and VI. 28, 29. Plutarch testifies that the problem of constructing a figure similar to a second figure and equal to a third was resolved by Pythagoras, and that he offered a sacrifice to the gods in gratitude for its solution. The same story of a sacrifice attaches to the discovery of the famous theorem of the three squares (Euclid I. 47), which in one particular case (the triangle with sides 3, 4, 5) was known to the Egyptians. Professor Loria inclines to the opinion that it was this property of the right-angled triangle which disclosed to Pythagoras the existence of irrational quantities; while Dr. Allman thinks this great discovery was rather owing to the problem: To cut a line in extreme and mean ratio. Such a mode of section must have been known to the brotherhood who adopted for their symbol of recognition the pentalpha or regular star pentagon.

J. S. Mackay.

CAJORI'S HISTORY OF MATHEMATICS.[1]

The extraordinary development, ramification, and specialisation which our generation sees in all science, but especially in the queen of sciences, mathematics, makes imperative a determined attempt to keep connected these new and growing fields.

For the individual, one way is so superior to all others that it must become a necessary part of real culture,

[1] *A History of Mathematics.* By Florian Cajori. Macmillan & Co., 1894.

namely, the way of scientific history, which masters real connection by living over again actual growth. In the sense of being a short general history of mathematics *carried up to the present day*, Cajori's book may be called unique in the world. Its greatest strength and originality lies in this modern part, but the whole treatment is conservative and pleasing.

It is a pity the author wholly misses one of the most interesting and instructive points in the history of mathematical thought, the question of the ratio of a piece of a curve to a piece of a straight. Did Archimedes consciously come down from Euclid's high critical standpoint when he made the paradoxical assumptions that the perimeter of an inscribed polygon is less, of a circumscribed greater, than the circle?

Again, the question of the parentage of our modern so-called Arabic numerals has been so extraordinarily fruitful, even in unexpected results about the political history of the Mohammedan peoples, that more than only a single sentence should be given to the true theory.

That the way of historical development is usually the most intelligible and interesting way of expounding any difficult part of a subject, is not only of highest pedagogic importance, but finds especial application in the teaching of the Calculus. But legend should be distinguished from history. The Introduction should not say: "After the class . . . tell them . . . how Pythagoras, jubilant over his great accomplishment, sacrificed a hecatomb to the Muses who inspired him," when we now know this not only unauthentic, but wholly out of accord with the character of the great master. Indeed, the pupil need only turn to page 21 of this very book to see the story discredited. GEORGE BRUCE HALSTED.

NOTES.

1. On an extension of VI. 3.

Theorem.—If AQ, AR be two chords of a triangle ABC equally inclined to the sides AB, AC; then

$$BQ.BR : CR.CQ :: BA^2 : CA^2 \text{ (Fig. 7).}$$

For $BQ : CR :: \Delta ABQ : \Delta ACR$ (vi. 1)

$:: AB.AQ : AC.AR$ (vi. 23 Cor.),

since $B\hat{A}Q = C\hat{A}R$.

So $BR : CQ :: \Delta ABR : \Delta ACQ$,

$:: AB.AR : AC.AQ$.

Compounding these ratios we have

$$BQ.BR : CR.CQ :: AB^2 : AC^2.$$

Cor.—If B, Q, R, C be given, the locus of A at which BQ, RC subtend equal angles is the same as the locus of points whose distances from B and C are in a given ratio—that is, a circle. R. W. GENESE.

2. A theorem connected with homography.

ABC is a triangle having AB = AC. AL is drawn parallel to CB. The angle LAB now rotates round A, its arms, in any position, meeting BC in Q, R. BR.QC is constant and $= BA^2$ (Fig. 8).

From construction $B\hat{A}R = L\hat{A}Q = A\hat{Q}C$,

and $A\hat{B}R = A\hat{C}Q$.

$\therefore \Delta ABR$ is similar to ΔQCA,

$\therefore AB : BR :: QC : CA$,

or $BR.QC = AB.CA = AB^2$.

(If $L\hat{A}C$, *i.e.* the supplement of LAB, be the rotating angle, we shall obtain $BQ.RC = AB^2$.)

Conversely, if B, C being fixed, Q, R be variable points on BC, such that $BR.QC = K^2$, then (provided $K > \frac{1}{2}BC$) we can find the point A so that QAR is a constant angle.

For the case in which $K < \frac{1}{2}BC$ a method (which is, too, applicable in all cases) of studying the relation of Q, R is to erect ordinates BH, CK each equal to K (and in the proper directions for the different cases); then (Fig. 9) HR, KQ are at right angles. Therefore, on the variation of Q, R, HR, KQ turn through equal angles.

R. W. GENESE.

3. On the fundamental equations in spherical trigonometry.

The following is suggested as an alternative investigation of the fifteen equations involving four elements of a spherical triangle:—

ABC (Fig. 10) is the triangle, O is the centre, EOK is the diametral plane perpendicular to the plane AOB. AC produced meets EOK in K. Then angle $mOn = A$, and taking radius as unity, $Om = \sin c$, $Bm = \cos c$, $On = \sin b$, $Cn = \cos b$.

$$\therefore mn^2 = \sin^2 b + \sin^2 c - 2 \sin b \sin c \cos A.$$

Now $(\text{chord of } BC)^2 = 4 \sin^2 \frac{a}{2} = mn^2 + (Cn - Bm)^2$,

$$\therefore 4\sin^2\frac{a}{2} = \sin^2 b + \sin^2 c - 2 \sin b \sin c \cos A + (\cos b - \cos c)^2,$$

which reduces to

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad \text{(i).}$$

This holds good for all values of sides and angles, whether greater or less than 90°.

The group of three such equations being found by symmetry, the group $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$ may be deduced as in the text books.

Join BK. Then from ΔABK, $\cos BK = \sin c \cos A$; and from ΔCBK, $\cos BK = \sin b \cos a - \cos b \sin a \cos C$; this second equation being found to hold good whether b is greater or less than 90°. Therefore in all cases

$$\sin c \cos A = \sin b \cos a - \cos b \sin a \cos C \qquad \text{(ii).}$$

$$\therefore \frac{\sin a \sin C}{\sin A} \cos A = \sin b \cos a - \cos b \sin a \cos C.$$

$$\therefore \cos b \cos C = \cot a \sin b - \cot A \sin C.$$

Interchanging a, b; A, B—

$$\text{Cos } a \cos C = \cot b \sin a - \cot B \sin C.$$

From this pair of equations, by symmetry, the entire group of six may be deduced.

From (ii) $\sin C \cos A = \sin B \cos a - \cos b \sin A \cos C$.

Interchanging, $\sin c \cos B = \sin A \cos b - \cos a \sin B \cos C$; eliminating $\cos b$ from these—

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

Fig 1.

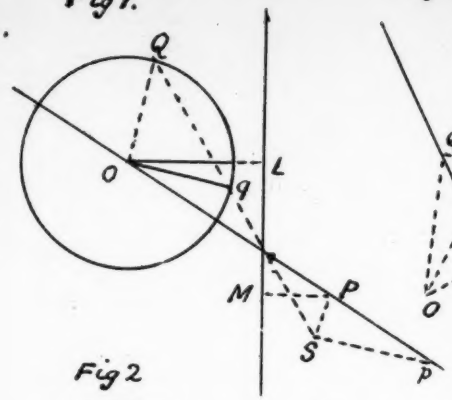
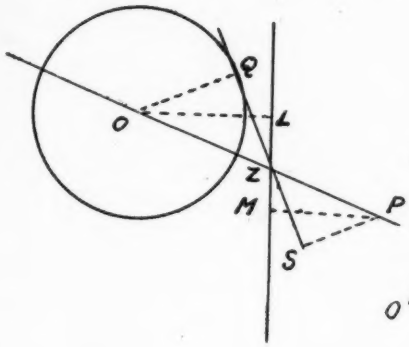


Fig 2



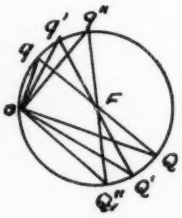


Fig 5

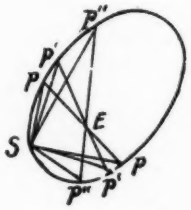


Fig 3

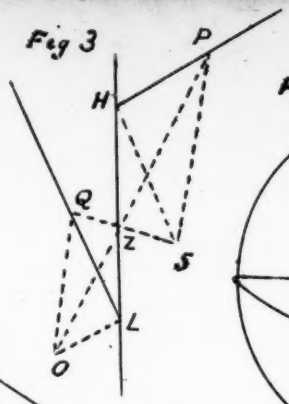


Fig 4

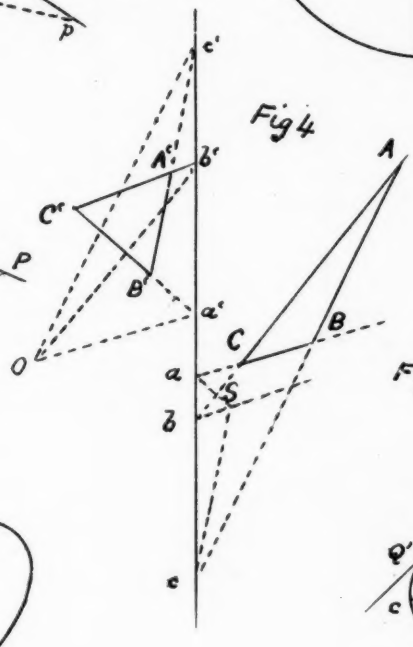


Fig 10

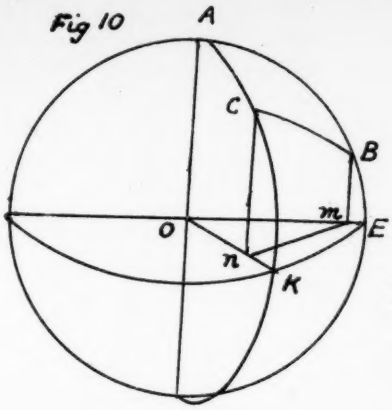


Fig 12

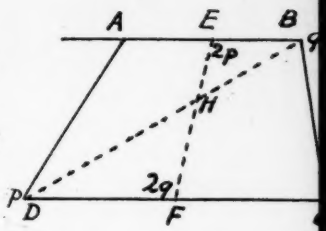


Fig 13

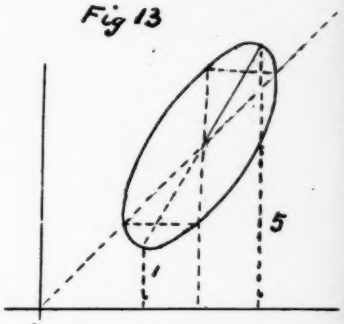


Fig 11

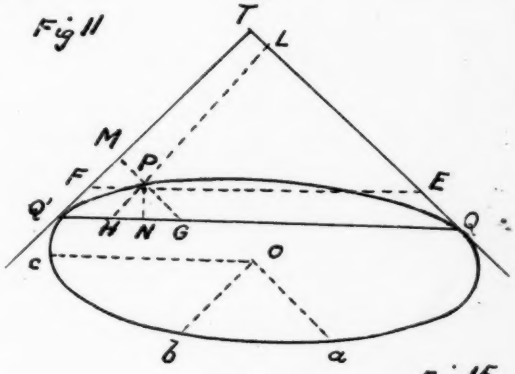


Fig 7

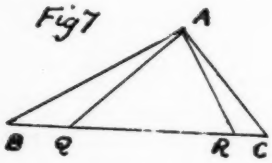


Fig 6

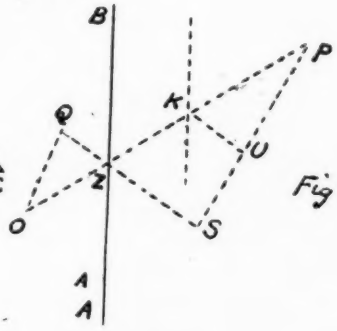


Fig 8

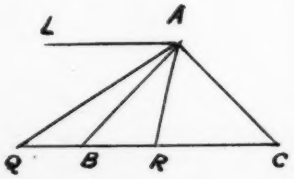


Fig 9

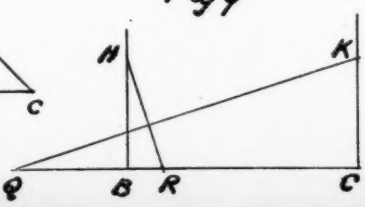


Fig 14

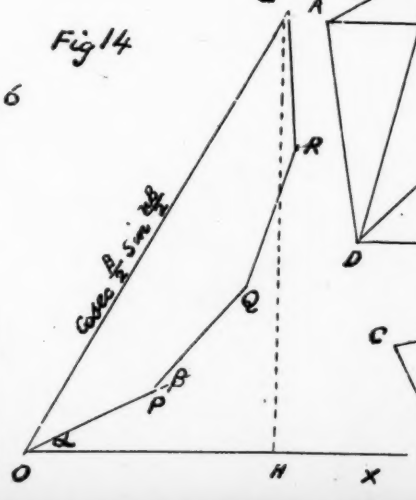
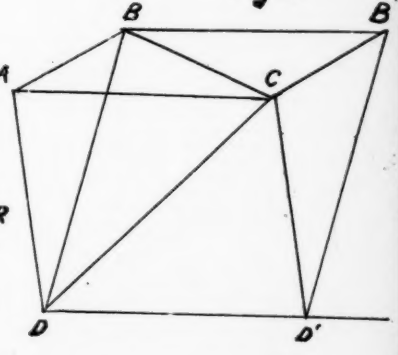


Fig 15



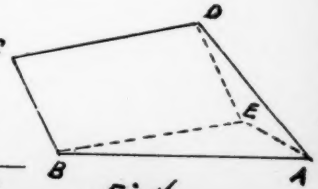


Fig 16

By symmetry the remaining group of three follows.

The properties of the polar triangle form a corollary to the above equations. G. HEPPEL.

4. Geometrical determination of K in the trilinear relation $\alpha\beta = K\gamma^2$.

TQ, TQ′ (Fig. 11) are two tangents to a conic, P any point on the curve. Through P and the centre O are drawn lines PG, PH, EPF, and Oa, Ob, Oc parallel to TQ, TQ′, QQ′ respectively. PL, PM, PN are perpendicular to TQ, TQ′, QQ′.

Then, since the diameter OT bisects QQ′, PP′, and EF, $\therefore$ EP′ = PF.

$$\therefore Oc^2 : Oa^2 = EP.EP' : EQ^2,$$
$$= EP.PF : EQ^2.$$

Since the right-angled triangles PEL, PGN are similar,

$$\therefore PE : PG(=EQ) = PL : PN.$$

Similarly, $PF : PH(=FQ') = PM : PN,$

$$\therefore PE.PF : EQ.FQ' = PL.PM : PN^2,$$

and $EQ : FQ' = PG : PH = TQ : TQ' = Oa : Ob,$

$$\therefore \frac{PL.PM}{PN^2} = \frac{PE.PF}{EQ.FQ'} = \frac{PE.PF}{EQ^2}\cdot\frac{EQ}{FQ'} = \frac{Oc^2}{Oa^2}\cdot\frac{Oa}{Ob},$$
$$= \frac{Oc^2}{Oa.Ob}.$$

It can easily be deduced from the above that in the trilinear relation—

$$\alpha\beta = K\gamma\delta,$$

if Oa, Ob, Oc, Od are the semi-diameters parallel to the four lines to which α, β, γ, δ are respectively perpendicular,

$$K = \frac{Oc.Od}{Oa.Ob}.$$ J. J. MILNE.

5. On the centroid of a trapezoid.

Let ABCD (Fig. 12) be a trapezoid whose parallel sides AB, DC contain p and q units of length respectively. Bisect AB, DC at E, F, and let BD, EF cut at H.

By VI. 1 triangle ABD : triangle BDC :: AB : DC,

$$:: p : q,$$
$$:: 3p : 3q.$$

The centroid of triangle ABD is that of masses $2p$ at E and p at D. The centroid of triangle BDC is that of masses $2q$ at F and q at B.

Now $EH : HF :: BH : HD,$

$$:: BE : DF,$$
$$:: p : q.$$

Hence masses p at D and q at B are equivalent to a mass $p+q$ at H, and therefore to masses p at F and q at E.

Hence the system has the centroid of masses $2p+q$ at E and $p+2q$ at F.

$\therefore$ it is at G on EF where $(2p+q)EG = (p+2q)FG$.

If we denote the centroids of triangles ABD, BDC by L and M, and adopt the notation of Möbius, the latter part of the demonstration would run thus—

$$(3p+3q)G = 3pL + 3qM,$$
$$= 2pE + pD + 2qF + qB,$$
$$= 2pE + 2qF + (p+q)H,$$
$$= 2pE + 2qF + qE + pF,$$
$$= (2p+q)E + (2q+p)F.$$

E. M. LANGLEY.

6. On a proof of XI. 4.

Euclid's own demonstration, being rather long, is now usually superseded either by the demonstration which Legendre gives in his *Elements of Geometry*, Bk. V. prop. 4, or by a third demonstration in which the perpendicular is produced to the other side of the plane. In Wilson's *Solid Geometry* this third method of proof is ascribed (erroneously) to Legendre, and in my own edition of Euclid to A. L. Crelle. The latter certainly gives it, without any hint of its authorship, in an article (dated 1834) in his *Journal*, vol. xlv. pp. 35, 36 (1853). He had however published it, along with the proofs of Euclid and Legendre, in his *Lehrbuch der Elemente der Geometrie*, vol. ii. p. 532 (1827), and added the remark: "This proof is by Cauchy."

The correctness of this ascription to Cauchy is confirmed by Lacroix, who gives the proof in his *Elements of Geometry*, § 196 (12th edition, 1822), with the note:

"This demonstration, of the same kind as that of Euclid but simpler, has been communicated to me by Mr. Cauchy, a very distinguished young geometer." J. S. MACKAY.

APPROXIMATIONS AND REDUCTIONS.

(Continued from No. 1, which contains fully-worked numerical examples of the application of the methods of "Practice" to such multipliers as those given below.)

"It is an abiding delusion of the opponent of decimals that he *will* suppose the decimalist to be under a contract never to use a common fraction."—*De Morgan.*

7. Degrees to radians ·01745329

$\frac{1}{60}(1+\frac{1}{20})(1-\frac{1}{400}-\frac{1}{60000}) = ·01745333.$

8. Minutes to radians ·000290888

(i) $·00029(1+\frac{3}{10000}) = ·00029087,$

(ii) $·0006(\frac{1}{2}-\frac{1}{70}-\frac{1}{1100}) = ·000290883.$

9. Seconds to radians ·00000484814

$·00001(\frac{1}{2}-\frac{1}{70}-\frac{1}{1100}) = ·00000484805.$

10. Revolutions p. min. to radians p. sec. ·104719755

$\frac{1}{10}(1+\frac{1}{20})(1-\frac{1}{400}-\frac{1}{60000}) = ·10472.$

11. Radians p. sec. to revolutions p. min. 9·5493

$10(1-\frac{3}{100}-\frac{3}{200}) = 10(1-\frac{1}{20}+\frac{1}{200}) = 9·55,$

i.e. multiply by 10 and subtract $4\frac{1}{2}$ per cent.

12. Miles to kilometres 1·60933

$2(1-\frac{1}{5})(1+\frac{1}{200}+\frac{1}{14000}) = 1·609\frac{3}{5},$

or $\frac{10}{6}(1-\frac{1}{30}-\frac{1}{10000}) = 1·609\frac{4}{9}.$

13. Kilometres to miles ·62138

(i) $(\frac{1}{2}+\frac{1}{8})(1-\frac{1}{20}-\frac{1}{14000}) = ·6214\frac{2}{7},$

(ii) $·6+\frac{1}{70}+\frac{1}{140} = ·6214\frac{2}{7}.$

Note that, approximately, 5 miles = 8 kilometres, or a mile and a quarter (*i.e.* 10 furlongs) = 2 kilometres, so that if our mile were increased by a quarter of its present length, the series *miles*, *furlongs*, *chains*, and *links* would

from a decimal system of measures just double the corresponding measures of the metric system.

At a rate of $3\frac{3}{4}$ miles per hour, a kilometre is a 10-minutes' walk, so that 6 kilometres an hour is a little easier than our 4 miles-an-hour standard of good walking.

14. Common to Napierian logarithms $2{\cdot}302585$

(i) $2{\cdot}3 + \frac{1}{400}(1 + \frac{1}{30}) = 2{\cdot}302581\frac{1}{3}$,

(ii) $2{\cdot}3 + \frac{1}{400}(1 + \frac{1{\cdot}02}{30}) = 2{\cdot}302585$.

15. Napierian to common logarithms ${\cdot}43429448$

(i) $\frac{1}{3} + \frac{1}{10} + \frac{1}{1000} - \frac{1}{300000} = {\cdot}4343$,

(ii) ${\cdot}4343(1 - \frac{1}{8000}) = {\cdot}43429457$,

(iii) $\frac{1}{3} + \frac{1}{10} + \frac{1}{1000} - \frac{1}{300000}(1 + \frac{1}{6}) = {\cdot}43429444\frac{4}{9}$,

(iv) $\frac{1}{3} + \frac{1}{9} - {\cdot}01015 = {\cdot}43429444\frac{4}{9}$.

EXAMPLE.—*Find* $\log_e 5$ *by* (14 ii), *and reconvert to common* logs *by* (15 iii) *and* (15 iv), *being given that* $\log_{10} 5 = {\cdot}6989700$.

·6989700	1·6094370	1·6094370
1·3979400	·5364790	·5364790
2096910	1609437	17882633
174742	160944	
5825		·71530533
29	·69903214	
		1609437
1·6094370	5365	16094
	894	8047
	·69896955	·69896955

Note.—The identity $(3 + \frac{1}{7})(1 - {\cdot}0004) = 3{\cdot}1416$ given in No. 1 had been previously noted and applied to numerical computation by Professor Genese in Milne's *Companion to Problem Papers*, p. 252.

SOLUTIONS OF EXAMINATION QUESTIONS.

The Editor will be glad to avail himself of the help of all classes of readers towards making this section of the Gazette *as useful as possible.* MATHEMATICAL TUTORS *are invited to send neat solutions;* STUDENTS *to call attention to classes of problems presenting exceptional difficulties, and* EXAMINERS *who sympathise with us to forward copies of their papers. The help of foreign readers is especially requested in obtaining copies of papers set in the public examinations of other countries.*

The Editor acknowledges with thanks the receipt of the following sets of papers:—

University of Edinburgh (Junior, Senior, and Advanced Honours).
Harvard University, 1893-94.
King's College, London, June 1894.
Vassar College, New York, June 1894.
Cambridge Higher Locals, June 1894.
West Point, Jan. 1894.

12. (i) *Find for real values of* x *the least and greatest values of* $x + \sqrt{6x - 7 - x^2}$. [Camb. Higher Locals.]

(ii) Find by algebra the greatest value of

$$x\sqrt{3} + \sqrt{1 - x^2}.$$

(i) $$x + \sqrt{6x - 7 - x^2} \quad \ldots \quad (a)$$
$$= x - 3 + \sqrt{2 - (x-3)^2} + 3.$$

Now $\{x - 3 + \sqrt{2 - (x-3)^2}\}^2 + \{x - 3 - \sqrt{2 - (x-3)^2}\}^2 = 4$.

Hence $\{x - 3 + \sqrt{2 - (x-3)^2}\}$ is numerically greatest when $x - 3 - \sqrt{2 - (x-3)^2} = 0$.

Hence its least and greatest values are -2 and $+2$.

Hence the least and greatest values of the given expression (a) are 1 and 5 respectively. (See Milne's *Companion to Problem Papers*, p. 9.)

The above example may be easily illustrated by a *graph.*

Putting $y = x + \sqrt{6x - 7 - x^2} \quad \ldots \quad (b)$

we obtain $(x-3)^2 + (y-x)^2 = 2$.

From which it is clear the graph is an ellipse (Fig. 13) which has $x - 3 = 0$, $y - x = 0$ for conjugate diameters, its equation referred to those diameters being $X^2/4 + Y^2/2 = 1$.

The points at which the tangents to the graph are parallel to the axis of x are, from simple geometrical reasons, those for which $X/\sqrt{2} = Y = \pm 1$, *i.e.* at which

$$x = 3 \pm X\sqrt{2} = 4 \text{ or } 2,$$
$$y = x + Y = 5 \text{ or } 1.$$

It is worthy of remark that the graphic method enables us to deal readily with the restricted case in which the radical in (a) is to be taken with the positive sign only, the minimum value of y being now $3 - \sqrt{2}$.

For an investigation of graphic methods see Chrystal's *Algebra*, ch. xv.

The "turning values" of (a) might also be found by considering the equation (b) as a quadratic in x. If x is to be real we find that $(y-5)(y-1)$ is not to be positive.

They may also be found by putting—

$$x - 3 = r\cos\phi, \sqrt{6x - 7 - x^2} = r\sin\phi,$$

whence $$y = \sqrt{2}(\sin\phi + \cos\phi) + 3,$$
$$= 2\sin(\phi + \tfrac{\pi}{4}) + 3.$$

Here, if we are restricted to the positive sign of the radical, we have only to deal with the range of values from $\phi = 0$ to $\phi = 180$.

(ii) Put $x\sqrt{3} = m$, $\sqrt{1 - x^2} = n$,

hence $$m^2 + 3n^2 = 3.$$

Now $$(m - 3n)^2 + 3(m + n)^2 = 4m^2 + 12n^2$$
$$= 12,$$

hence $m + n$ is greatest when $m = 3n$, which gives $m + n = 2$.

We have here a particular case of the identity

$$(pm - qn)^2 + pq(m + n)^2 = (p + q)(pm^2 + qn^2),$$

whence p and q, being positive constants,

(a) $m + n$ is greatest for a given value of $pm^2 + qn^2$,

(β) $pm^2 + qn^2$ is least for a given value of $m + n$

if $pm = qn$.

13. PQ *is a focal chord and* R *any point on the conic. If* PR, QR *meet the directrix in* T *and* U, *show that* XT . XU *is constant.* [Univ. Ed. Sen. Math.]

Revert the conic into a circle in Boscovich's manner. Then ST, SU are parallels to the arms of the angle in a semicircle. Hence TSU is a right angle and $XT \cdot XU = SX^2$.

The question was given as a rider to one on Boscovich's construction.

14. *Apply the theory of orthogonal projection to sum the series*—

$$\cos\alpha + \cos(\alpha+\beta) \ . \ . \ . \ + \cos(\alpha + \overline{n-1}\beta$$

[*Ibid.*]

Draw a broken line OPQR . . . Z (Fig. 14) such that OP, PQ, QR, etc. each = unity, and angles OPQ, PQR, etc. each $= \pi - \beta$.

Then each of these is a chord of a circle and subtends an angle $= \beta$ at the centre.

Hence (i) diameter of circle = cosec $\frac{1}{2}\beta$.

(ii) $OZ = \text{cosec}\ \frac{1}{2}\beta \sin\frac{1}{2}n\beta$.

(iii) angle $POZ = \frac{1}{2}(n-1)\beta$.

Draw OX, making angle $POX = \alpha$

and $\therefore$,, ,, $XOZ = \alpha + \frac{1}{2}(n-1)\beta$.

Also OP, PQ, QR . . . make angles α, $\alpha+\beta$, $\alpha+2\beta$ . . . with OX, hence $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) +$ etc.—

= sum of projections of OP, PQ, QR, etc. on OX,

= projection of OZ on OX,

$= \text{cosec}\ \frac{1}{2}\beta \sin\frac{1}{2}n\beta \cos\{\alpha + \frac{1}{2}(n-1)\beta\}$.

If OY be drawn at right angles to OX, $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) +$ etc.—

= sum of projections of OP, PQ, QR, etc., on OY,

= projection of OZ on OY,

$= \text{cosec}\ \frac{1}{2}\beta \sin\frac{1}{2}n\beta \sin\{\alpha + \frac{1}{2}(n-1)\beta\}$.

15. (i) *If in a quadrilateral* ABCD *the part* ABD *be translated parallel to* AC *till* A *comes to* C *into the position* CB′D′, *show that* BDD′B′ *is a parallelogram.*

(ii) *Given of a quadrilateral its diagonals and area and two opposite sides, construct it.* [Ed. Jun. Math.]

The demonstration of (i) is simple. In Petersen's *Methods and Theories* (Sampson, Low, and Co.) it is pointed out (p. 48) that the above figures have the following additional properties:—

(i) The lines CB′, CD′, CD, CB = the sides of the original quadrilateral.

(ii) The angles BCB′, B′CD′, D′CD, DCB = the angles of the original quadrilateral.

(iii) The sides of the parallelogram = the diagonals of ABCD, and are inclined at the same angles.

(iv) The area of the parallelogram is twice that of ABCD.

The method of investigation here adopted by Petersen is called *Parallel Translation.* He remarks that it "will specially be applicable where two lines and the angle which they form are known; by translation of one line till its extremity coincides with that of the other we obtain a triangle which can be drawn at once." In the case of the quadrilateral ABCD it is especially applicable "if the diagonals and their enclosed angle be given; in this case we are able to draw the parallelogram directly, and the problem is therefore reduced to the determination of point C." Hence the following construction:—

(ii) Make a parallelogram BDD′B′ double the given area of the required quadrilateral, and whose sides BB′, B′D′ are equal to the given diagonals (Fig. 15).

With centres B and D′ and radii = the given lengths of BC, DA, describe arcs intersecting in C. Through C draw CA equal and parallel to B′B and in the same sense. Then ABCD is the required quadrilateral.

As another instance of the value of the general method we take the following simple problem.

(iii) *Construct a quadrilateral of which the lengths of the sides are given and the inclination of one pair of opposite sides.* [Gonville and Caius Entr. Schol.]

Suppose BC and DA to be the sides whose inclination is known, and that in the required quadrilateral ABCD, BC were, in accordance with Petersen's directions, translated parallel to CD until it came to the DE. Then the triangle ADE can be constructed, since we know the two sides ED, DA, and the included angle. Hence also the triangle ABE can be found, since we are given the other two sides AB, BE (BE = CD). We have now only to complete the parallelogram on DEBC (Fig. 16).

16. *Prove that the greatest coefficients in the expansion of the trinomial*

$$(x+y+z)^{3n+1}$$

have the value $(3n+1)\,!/\,(n\,!)^3(n+1)$.

Any coefficient is known to be of the form

$$(3n+1)!/r\,!\,s\,!\,t\,!$$

where $r+s+t = 3n+1$ (a). In the investigation of the greatest coefficient in a binomial expansion it is shown that *for a given value of* $r+s$ the product $r\,!\ s\,!$ is least when r and s differ by 0 or 1, according as $r+s$ is even or odd. Hence any two of the numbers r, s, t in a maximum coefficient can only differ by 0 or 1. Hence they must be of the forms (i) p, p, p, (ii) $p, p, p+1$, (iii) $p, p+1, p+1$. But (i) and (iii) are inadmissible by the condition (a).

Hence the greatest coefficient is $(3n+1)!/\ n!\ n!\ (n+1)!$

17. *Find the area enclosed by* 200 *hurdles placed so as to form a regular polygon of* 200 *sides, the length of each hurdle being* 6 *ft.* [W. Jan. 1894.]

As this question occurs in one of the obligatory papers, candidates were probably to be content with treating the 6 ft.-chord of the circumscribing circle as being sensibly equal to its arc. With this assumption we have approximately—

Area $= \frac{1}{2}$ rad. $\times$ circumf.	360,000
$= \frac{1}{2} \times \frac{6 \times 100}{\pi} \times 1200$	120,000
	3,600
$= \frac{360{,}000}{\pi}$	1,800
	114,600
= 114,600 sq. ft.	

See article on *Approximations and Reductions* in No. 1.

If we take a closer approximation $\theta \div \tan\theta = 1 - \frac{\theta^2}{3}$ we get—

$$\text{Area} = \frac{360{,}000}{\pi}\left\{1 - \tfrac{1}{3}\left(\frac{\pi}{400}\right)^2\right\}$$

$$= \frac{360{,}000}{\pi}(1 - \cdot 00002),$$

which shows that the error made by taking the area as $360{,}000 \div \pi$ is relatively less than made in taking the reciprocal of π as $\frac{1}{3} - (\frac{1}{100} + \frac{1}{100}) = \cdot 3183\frac{1}{3}$.

QUESTIONS FOR SOLUTION BY STUDENTS.

29. A blindfolded man continually walks straight a constant distance and then wheels left through a constant angle. Prove by elementary geometry that if the angle be a submultiple of four right angles he will return to the starting-point. Prof. R. W. GENESE.

30. Defining the parabola as the path of a projectile under a force constant in direction and magnitude, show that the locus of the mid-points of a set of parallel chords is a straight line. H. D. DRURY, M.A.

31. If $a^2 + b^2 = c^2$, then $(2a + b + 2c)^2 + (a + 2b + 2c)^2 = (2a + 2b + 3c)^2$. Hence show how to find an indefinite number of rational right-angled triangles whose legs differ by unity. Prof. ARTEMAS MARTIN.

32. If a system of coaxal circles having real limiting points L, L′ be inverted from any point E in their line of centres, the radius of inversion being $\sqrt{(EL \cdot EL')}$, then each circle of the system inverts into another circle of the system, and the circle through E inverts into the radical axis of the system, and is the circle of similitude of each pair of mutually inverse circles. H. D. ELLIS, M.A.

33. Solve the equation—

$$\frac{x + \sqrt{1-x^2}}{1 + x + \sqrt{1-x^2}} = \frac{x - \sqrt{1-x^2}}{1 - x + \sqrt{1-x^2}}$$

by trigonometry. EDITOR.

34. The pth term of an A. P. is k, the qth term l, find the $(p+q)$th term and account *geometrically* for the result obtained. EDITOR.

35. In a certain examination every candidate must take either Latin or Mathematics, or both. 79·4 per cent of the candidates took Latin, 89·6 per cent took Mathematics. If there were 1500 candidates altogether, what are the greatest and the least numbers that could have taken both Latin and Mathematics? St. Andrews Loc. Ex. 1892.

36. A plane cuts OA, the axis of a right circular cylinder, at right angles in O; B′OB and D′OD are two diameters of the cylinder at right angles to each other in the given plane. The given plane revolves round B′OB. Find the locus of either of the foci of the variable ellipse thus generated; and show that DT, a straight line through D parallel to the axis of the cylinder, is an asymptote to the curve. Lieut.-Col. H. W. L. HIME.

37. Deduce the expression for the bisector of an angle of a triangle from a known theorem on the resultant of two forces. EDITOR.

38. Find by an elementary method the maximum value of $\cos\theta\{\sin\theta + \sqrt{(\sin^2\theta + \sin^2\alpha)}\}$. EDITOR.

39. Find a number of two digits equal to the square of the tens' digit together with the square of the sum of its digits. EDITOR.

[*The Editor would be glad to receive contributions to this column. Solutions to questions* 19, 20, *and* 21 *by "Flamingo," to question* 4 *by the Rev. J. J. Milne, and to question* 26 *by Prof. Hudson have been received.*]

BOOKS AND MAGAZINES RECEIVED.

Geometrical Conics. By the Rev. J. J. MILNE, M.A., and R. F. Davis, M.A. (Macmillan and Co.)

Geometrical Conics. By C. SMITH, M.A. (Macmillan and Co.)

Integral Calculus for Beginners. By J. EDWARDS, M.A. (Macmillan and Co.)

Our space does not admit of a review of the above important new text-books, but we hope to give them due consideration in No. 4.

Geometrical Exercises in Paper Folding. By T. SUNDARA ROW, B.A. An attempt to develop "Kindergarten Gift No. VIII." into a useful course of geometry. The author has sought, in our opinion with success, "not only to aid the teaching of geometry in schools and colleges, but also to afford mathematical recreation to young and old." The book should be in the hands of all those who have to introduce geometrical ideas to young pupils. We commend it especially for use in kindergarten training colleges for teachers.

The Mathematical Magazine. Vol. II., Nos. 1-8.

The American Mathematical Monthly. Vol. I., Nos. 4-8.

Journal de Mathematiques Élémentaires. Nos. 7-10, 1894.

Periodico di Matematica. Sept.-Oct. 1894.

Indian Engineering. Vol. xvi. No. 7.

THE GENERAL MEETING of the ASSOCIATION for the IMPROVEMENT of GEOMETRICAL TEACHING is to be held on *Saturday, January* 19, 1895. For particulars see page 3 of cover. All interested in the work of the Association are invited to be present.

Fig. i

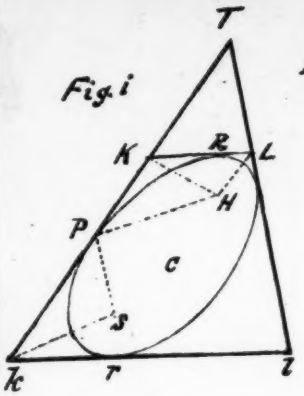


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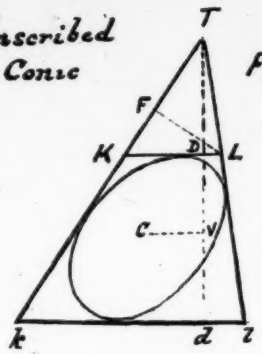
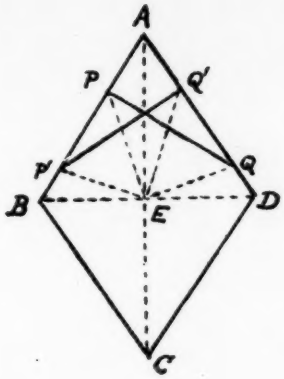
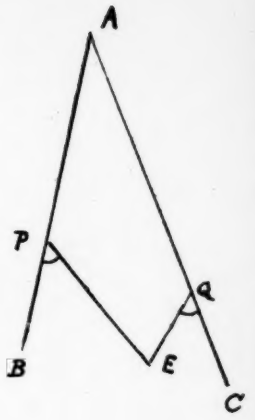
Fig. ii



Solutions

9.





Note on Syllabus

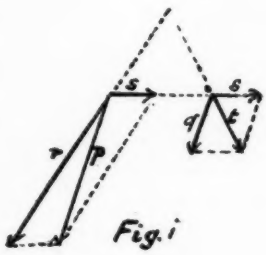


Fig. i

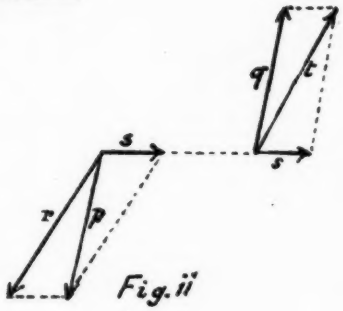
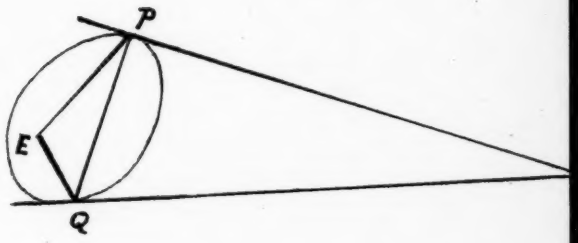
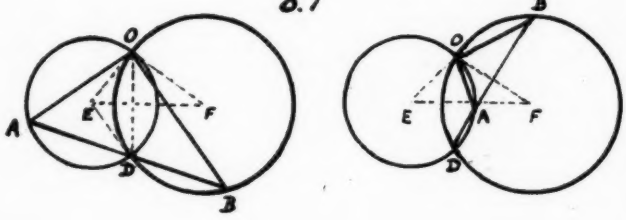


Fig. ii

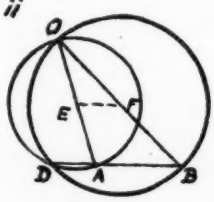


Solutions

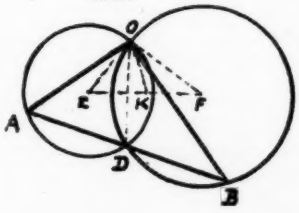
8. i



8. ii



8. iii



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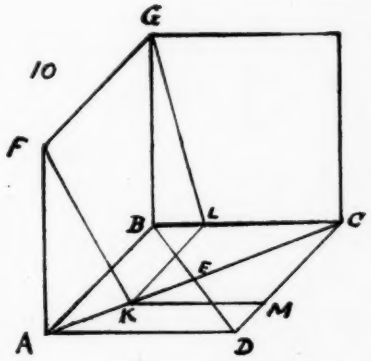
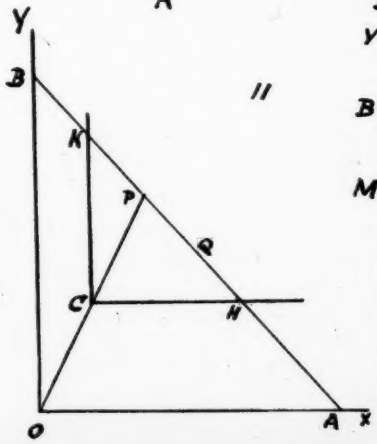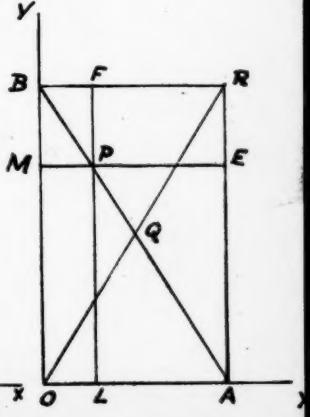
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